WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2022-23

MTMACOR05T-MATHEMATICS (CC5)

THEORY OF REAL FUNCTIONS

Time Allotted: 2 Hours — Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Prove that $\lim\limits_{x\to 0} x^{3/2} = 0$.

(b) Show that $\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist.

(c) Determine the value of a so that

$$f(x) = \begin{cases} x+1 & ; \quad x \le 1 \\ 3-ax^2 & ; \quad x > 1 \end{cases}$$

is continuous at $x = 1$.

(d) Give an example of two functions $f, g : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, which are not continuous at a point $c \in I$, but $f + g : I \to \mathbb{R}$ is continuous at c.

(e) Show that the function $f : [1, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x ; & x \in [1, 2] \cap \mathbb{Q} \\ -x ; & x \in [1, 2] - \mathbb{Q} \end{cases}$$

is discontinuous at every point of $[1, 2]$.

(f) Examine the differentiability of $f(x)$ at $x = 0$ and $x = 1$ where

$$f(x) = \begin{cases} 1-x^2, & -1 \le x < 0 \\ x^2+x+1, & 0 \le x < 1 \\ x^3+2, & 1 \le x \le 2 \end{cases}$$

(g) Examine validity of Rolle's theorem for the function

$$f(x) = \sin x \cos x, \qquad x \in \left[0, \frac{\pi}{2}\right]$$

Also, verify the conclusion of Rolle's theorem for this function, if possible.

(h) Examine the validity of the hypothesis and conclusion of Lagrange's mean value theorem for the following function:

$$f(x)=x(x-1)(x-2) \quad x\in\left[0, \frac{1}{2}\right]$$

(i) Find the maximum value of the function

$$y=1+2\sin x+3\cos^2 x, \quad 0\le x\le\frac{\pi}{2}.$$

2. (a) Let $f:D\to\mathbb{R}$, where $D\subseteq\mathbb{R}$ and let $\lim_{x\to a} f(x)=l$. Show that there is a neighborhood N of a so that f is bounded on $(N-\{a\})\cap D$. 4

(b) Show that 4

$$\lim_{x\to\infty}\frac{x+[x]}{x^2}=0$$

where $[x]$ denotes the integral part of x for any $x\in\mathbb{R}$.

3. (a) Let $f:I\to\mathbb{R}$ and $g:J\to\mathbb{R}$ be such that Image $f\subseteq J$, f is continuous at $a\in I$ and g is continuous at $f(a)\in J$. Show that the composition $g\circ f:I\to\mathbb{R}$ is continuous at a. 4

(b) Let $f:I\to\mathbb{R}$ be a function continuous at $c\in I$, where I is an interval in $\mathbb{R}$. Let f take both positive and negative values in each neighborhood of c. Show that $f(c)=0$. 4

4. (a) Let $f:D\to R$ $(D\subset R)$ be a function and c be a limit point of D. Let $l\in R$, then prove that $\lim_{x\to c} f(x)=l$ if and only if for every sequence $\{x_n\}$ in $D-\{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l. 2+3

(b) Using the above theorem prove that $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist. 3

5. (a) Prove that every continuous function f on a closed and bounded interval $[a, b]$ is bounded and there exists a point $c\in[a, b]$ such that 3+2

$$f(c)=\sup_{x\in[a, b]} f(x)$$

(b) Let $I=[a, b]$ be a closed and bounded interval and $f:[a, b]\to R$ be continuous on I, then prove that $f(I)=\{f(x):x\in I\}$ is a closed and bounded interval. 3

6. (a) Is a function $f:I\to R$ which is uniformly continuous on I, continuous on I? Give reason. 3+1

Under what condition a continuous function $f:I\to R$ will be uniformly continuous?

(b) If $f: D \to R\ (D \subset R)$ be uniformly continuous on D and $\{x_n\}$ be a Cauchy sequence in D, then prove that $\{f(x_n)\}$ is a Cauchy sequence in R. Using this, prove that $f(x) = \frac{1}{x}$, is not uniformly continuous on $(0, 1)$. 2+2

7. If $f: I \to R$ is differentiable at $c \in I$, then prove that f is increasing at $x = c$ if $f'(c) > 0$. 2+2+4

Is the condition necessary for a function to be increasing at a point? Give reason.

Use this result to prove that

$$\frac{x}{1+x} < \log(1+x) < x \quad \text{for all } x > 0.$$

8. (a) Let a function f be derivable in some closed and bounded interval $[a, b]$ and $k \in \mathbb{R}$ with $f'(a) < k < f'(b)$. Then prove that there exists at least one point $c \in (a, b)$ such that $f'(c) = k$. 4

(b) If $\varphi(x) = f(x) + f(1-x)$ and $f''(x) < 0$ in $[0, 1]$. Then show that φ is monotone increasing in $[0, \frac{1}{2})$ and monotone decreasing in $(\frac{1}{2}, 1]$. 4

9. (a) State Rolle's Theorem. If $p(x)$ be a polynomial of degree > 1, prove that there is a root of $p'(x) + kp(x) = 0$, k being a real constant, between two distinct roots of $p(x) = 0$. 2+3

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as follow: 3

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is derivable at $x = 0$ but derived function is not continuous at $x = 0$.

10.(a) If a function f be such that $f^{(n)}(a)$ exists and M be defined as follows: 4

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}M$$

Then show that $M \to f^{(n)}(a)$ as $h \to 0^+$.

(b) Show that for the function $f(x) = \frac{(2x-1)(x-8)}{x^2 - 5x + 4}$ the minimum value is greater than the maximum value. 4

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